

CCFU Proof 18

Uniqueness of C_2 Among ± 1 Recurrences

Given. All second-order recurrences $x_{n+1} = a x_n + b x_{n-1}$ with $a, b \in \{+1, -1\}$.

Claim. C_2 (i.e., $a = b = 1$) is, up to isomorphism, the unique such recurrence with real hyperbolic roots and a reciprocal magnitude pair.

Proof. There are exactly four cases.

Case 1: $(a, b) = (1, 1)$.

$$\lambda^2 - \lambda - 1 = 0, \quad \Delta = 5, \quad \text{roots: } \varphi, -1/\varphi.$$

Real roots, reciprocal magnitudes $\{\varphi, 1/\varphi\}$. ✓

Case 2: $(a, b) = (-1, 1)$.

$$\lambda^2 + \lambda - 1 = 0, \quad \Delta = 5, \quad \text{roots: } 1/\varphi, -\varphi.$$

Isomorphic to Case 1 via $x_n \mapsto (-1)^n x_n$. Same magnitudes, same chain. ✓

Case 3: $(a, b) = (1, -1)$.

$$\lambda^2 - \lambda + 1 = 0, \quad \Delta = -3, \quad \text{complex roots, } |\lambda| = 1.$$

No real reciprocal pair. Chain breaks at step 1. ✗

Case 4: $(a, b) = (-1, -1)$.

$$\lambda^2 + \lambda + 1 = 0, \quad \Delta = -3, \quad \text{complex roots, } |\lambda| = 1.$$

No real reciprocal pair. Chain breaks at step 1. ✗

Conclusion. Only Cases 1 and 2 yield $\Delta = 5$, real hyperbolic roots, and a reciprocal magnitude pair. These two are isomorphic. Cases 3 and 4 give $\Delta = -3$ with complex roots on the unit circle; no growth/decay split, no real reciprocal pair, no factor completion.

Therefore C_2 is, up to isomorphism, the unique minimal parameter-free second-order recurrence whose spectral chain reaches $\text{sig}(3, 2)$. ■

[No dependencies. Self-contained.]